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Causal Symmetry Transformations and their Representations by Semigroups

Arno R. Bohm,[1] Yoshihiro Sato,[1] and Piotr Kielanowski[2,3]

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If one distinguishes between states and observables in quantum theory one obtains from causality arguments that the quantum theoretical symmetry transformations of non relativistic and relativistic space time do not form a group but a semigroup into the forward light cone. These semigroup representations describe resonances and decaying states.

KEY WORDS: resonances; rigged Hilbert space; semigroup.

1. INTRODUCTION

Groups and their unitary representations have been the favored tools for the derivation of symmetry properties since the advent of quantum mechanics. They are used to simplify and reduce the number of matrix elements of important observables like, e.g., the matrix elements of the S-operator. But in general one does not need to assume for this purpose the existence of a group of symmetry transformations, a continuous subset like, e.g., a semigroup of transformations would do the job.

The main support argument in favor of a unitary group representation does not come from a physical argument, but—most likely—originates from the Hilbert space axiom of quantum mechanics (von Neumann, 1955). As a consequence of the Hilbert space boundary condition the dynamical equations (Schrödinger or Heisenberg equation) integrate to the unitary representation of the time translations (Wigner, 1939). From here it was a straightforward generalization to the unitary group representations of the Poincaré transformations (Wigner, 1939) and to the domineering role of group representations in quantum theory.

It is remarkable that semigroup representations of Poincaré transformations $\mathcal{P}$ were also mentioned long time ago in connection with decaying particles

[1] Department of Physics, The University of Texas at Austin, Austin, TX 78712.
[2] Departamento de Física, Cinvestav, Ap. Postal 14-740, 07000 Mexico D.F., Mexico.
[3] To whom correspondence should be addressed; e-mail: kiel@fis.cinvestav.mx.

(Schulman, 1970). The semigroup representation could have been used in the formulation of causality, specifically of "Einstein causality" (Born probabilities can only propagate with a velocity smaller than the velocity of light) (Bohm *et al.*, 2002). But instead relativistic quantum field theory (Streater and Wightman, 1986) formulated causality in the form of local commutativity. The local commutativity was imposed upon the theory together with axioms, without making sure that there is no conflict between these different axioms. One of the other axioms was the unitarity of the Poincaré transformations with asymptotic completeness; the Poincaré semigroup representations (Schulman, 1970) were ignored.

Semigroup representations for time evolution and space-time symmetry emerged again in connection with a theory of resonance scattering and decay. Resonances in scattering experiments or decaying states in decay experiments are connected with an asymmetric or "irreversible" time evolution. Thus they require a time asymmetric quantum theory, whereas the quantum mechanics based on the Hilbert space axiom is a theory with reversible time evolution. Using Hilbert space mathematics including Dirac kets defined as Schwartz space functions, one runs into contradictions. Therefore, in the heuristic treatment of scattering theory one just ignored the mathematical subtleties. One worked with mathematically undefined Lippmann–Schwinger kets (Brenig and Hagg, 1959; Gell-Mann and Goldberger, 1953; Lippmann and Schwinger, 1950; Newton, 1982), used $\pm i\epsilon$ to formulate incoming and outgoing boundary conditions, restricted by fiat the time in e^{iHt} to $t \geq 0$ (Gell-Mann and Hartle, 1994, 1995), and for decaying states one postulated purely outgoing boundary conditions (Peierls, 1938, 1995) undisturbed by the fact that this was in conflict with the unitary group evolution $-\infty < t < \infty$.

In order to relate the S-matrix pole and its scattering amplitude with Breit–Wigner energy distribution (which defines a resonance) to exponentially evolving Gamow vector (by which we define a decaying state), one had to replace the Hilbert space axiom by a Hardy space axiom (Bohm *et al.*, 1997; Bohm, 1981). The Hilbert space axiom postulates a one-to-one correspondence between energy wave functions for states as well as for observables to the same Hilbert space (Lebesgue square integrable functions). The Hardy space axiom uses Hardy functions analytic in the lower half plane as energy wave function of states and Hardy functions analytic in the upper half plane as energy wave functions of observables. This provided a mathematical distinction between a state (defined by a preparation apparatus) and an observable (defined by a detector). It leads to semigroup representations for time translations and to semigroup representations for symmetry transformations of the relativistic space time. Can we accept such a modification of quantum mechanics in view of some famous theorems (Bargmann, 1964; Wigner, 1952) to the contrary?

In this paper we shall argue that quantum mechanical space-time transformation are indeed given by semigroups. In a subsequent paper we shall discuss the observation of the main aspect, by which the semigroup differ from the unitary group evolution, the semigroup time $t_0 = 0$ (Bohm *et al.*, 1997; Bohm, 1981).

2. FOUNDATIONS OF QUANTUM MECHANICS—GROUPS AND SEMIGROUPS

In quantum theory one speaks of states and of observables. States are described by density operators ρ, W, or by vectors φ when the state is a pure states. Observables are described by operators $A(= A^\dagger)$, by positive operators Λ, or also by vectors ψ if $\Lambda = |\psi\rangle\langle\psi|$.

A state W (for instance the in-states φ^+ of a scattering experiment) is prepared by the preparation apparatus (e.g., an accelerator). An observable Λ (out-observables ψ^- or "out-state") is registered by the registration apparatus (e.g., a detector).

The measured quantities are interpreted as the probabilities $\mathcal{P}_W(\Lambda(t))$ to measure observable Λ in the state W or the observable ψ in the state φ. They are calculated in the theory as Born probabilities and are measured as ratios of large numbers of detector counts:

$$\mathcal{P}_W(\Lambda(t)) \equiv Tr(\Lambda(t)W_0) = Tr(\Lambda_0 W(t)) \approx N(t)/N, \tag{1a}$$

$$|\langle\psi^-(t)|\varphi^+\rangle|^2 = |\langle\psi^-|\varphi^+(t)\rangle|^2 \tag{1b}$$

The time evolution of the Born probabilities are described in the Heisenberg picture with time dependent observable $\psi^-(t)$ and the state φ^+ fixed at a time $t = 0$, or in the Schrödinger picture with a time-dependent state $\varphi^+(t)$ and ψ^- fixed at $t = 0$. The dynamical equations are the Heisenberg equation for observables or Schrödinger equation for states:

$$i\hbar\frac{\partial}{\partial t}\psi^-(t) = -H\psi^-(t) \quad \text{or} \quad i\hbar\frac{\partial}{\partial t}\varphi^+(t) = H\varphi^+(t) \tag{2}$$

To solve the dynamical differential equation one has to choose the *boundary conditions*, this is done in traditional quantum mechanics by the Hilbert space axiom:

$$\text{set of in-states } \{\varphi^+\} = \text{set of out-observables } \{\psi^-\} = \mathcal{H} \tag{3}$$

The solution of the dynamical equations under the Hilbert space boundary condition (3) is, for observables

$$\psi^-(t) = e^{iHt}\psi^-, \quad \psi^- \in \mathcal{H}, \quad \text{with } -\infty < t < \infty, \tag{4}$$

and for states

$$\varphi^+(t) = e^{-iHt}\varphi^+, \quad \varphi^+ \in \mathcal{H}, \quad \text{with } -\infty < t < \infty. \tag{5}$$

The time translation (of e.g. the observable ψ^-) is given by the unitary group $\mathcal{U}(t) \equiv e^{i\bar{H}t}$, $-\infty < t < +\infty$. This means for every translation exists also and inverse which is given by $(\mathcal{U}(t))^{-1} = \mathcal{U}(-t) = e^{-i\bar{H}t}$.

A set of elements, fulfilling the same requirements as a group, except that not every element has an inverse, is called a semigroup. Specifically a one parameter semigroup of time evolution which is a solution of the Schrödinger equation (2) has also the same form as (5):

$$\mathcal{U}^{\times}(t) = e^{-iH^{\times}t} \text{ but with } t \text{ restricted to } 0 \leq t < \infty. \tag{6}$$

Since the theory of resonances and decaying states has problems with axiom (3), basically because the vectors with exponential time evolution as well as the Lippmann–Schwinger in- and out- kets are not in $\mathcal{H}$, and are also not Schwartz space functionals like Dirac kets, we changed the boundary condition (3) to get the analyticity property of the energy wave functions required to include Gamow states and Lippmann–Schwinger kets (Bohm *et al.*, 1997; Bohm, 1981). This led to a semigroup time evolution like Eq. (6). If we consider not only time translations but the group of all symmetry transformations of non-relativistic space-time, then we obtain in place of Eq. (6) the quantum mechanical Galilei group. And if we consider all (continuous) symmetry transformations of the relativistic space-time then we obtain the Poincaré *semigroup* representations in the forward light cone. This is in contrast to the prevailing viewpoint that the *unitary group* representations represent the quantum physical systems. In the present paper we give heuristic arguments in favor of the semigroup representation.

3. SYMMETRY GROUPS AND THEIR QUANTUM PHYSICAL REPRESENTATIONS

The symmetry transformations $\mathcal{T}$ of non-relativistic space-time are given by the Galilei group

$$\mathcal{G} = \{(R, \boldsymbol{x}, \boldsymbol{v}, t)\}, \tag{7a}$$

$$\boldsymbol{x}_0 \to \boldsymbol{x}' = R\boldsymbol{x}_0, \qquad \boldsymbol{x}_0 \to \boldsymbol{x}' = \boldsymbol{x}_0 + \boldsymbol{x}, \qquad \boldsymbol{x}_0 \to \boldsymbol{x}' = \boldsymbol{x}_0 + \boldsymbol{v}t, \tag{7b}$$

$$t_0 \to t' = t_0 + t. \tag{7c}$$

The symmetry transformations $\mathcal{T}$ of relativistic space-time are given by the Poincaré group

$$\mathcal{P} = \{(\Lambda, x)\}, \quad x = (t, \boldsymbol{x}) \tag{8a}$$

$$x_0 \to x' = x + \Lambda x_0, \quad \Lambda^T g \Lambda = g, \quad \det(\Lambda) = 1, \quad x \in R_4, \tag{8b}$$

$$(\Lambda, x)(\Lambda', x') = (\Lambda\Lambda', x + \Lambda x') \tag{8c}$$

In quantum physics one does not deal with space-time transformations themselves but with representations thereof

$$(\Lambda, x) \longrightarrow \mathcal{U}(\Lambda, x) \tag{9}$$

where $\mathcal{U}(\Lambda, x)$ are operators in the linear space of states $\{\varphi\}$ (or of the set of density operators W).

Alternatively, $\mathcal{U}(\Lambda, x)$ can be operators in the space of observables $\{\psi\}$ (or of observable operators A, $|\psi\rangle\langle\psi|$).

A state φ is prepared by a preparation apparatus (e.g., an accelerator), an observable $|\psi\rangle\langle\psi|$ is detected by a registration apparatus (detector).

Symmetry transformations are transformations of an observable relative to a state or they are transformations of a state relative to and observable in the opposite direction. It does not matter whether we do the one or the other.

To illustrate this we consider the observable

$$|\psi\rangle\langle\psi| = \int_{\Delta^3 x} |\boldsymbol{x}_i\rangle\langle\boldsymbol{x}_i|\, d^3x_i = |\boldsymbol{x}_i\rangle\langle\boldsymbol{x}_i|\Delta^3 x, \tag{10}$$

for the detector position in Fig. 1(b), where $\Delta^3 x$ is the volume of the (ideal) detector. The preparation apparatus prepares the state φ in Fig. 1(a). The counting rate of the detector (number of clicks per second) at the position $\boldsymbol{x}_i$ of the detector of Fig. 1(b) for the state φ prepared by the apparatus of Fig. 1(a) is given by the Born probability:

$$\langle\varphi|\psi\rangle\langle\psi|\varphi\rangle = |\langle\boldsymbol{x}_i|\varphi\rangle|^2\Delta^3 x. \tag{11}$$

This configuration is not depicted by a figure.

Now we transform the detector into $|\boldsymbol{x}_i^T\rangle\langle\boldsymbol{x}_i^T|\Delta^3 x_i^T$ around the position x_i^T to obtain the configuration of Fig. 1(a). The Born probability is then $|\langle\boldsymbol{x}_i^T|\varphi\rangle|^2\Delta^3 x_i^T$. Alternatively we can leave the detector where it was in Fig. 1(b) and transform the state into the opposite direction to $\varphi^{T^{-1}}$. The Born probability is the $|\langle\boldsymbol{x}_i|\varphi^{T^{-1}}\rangle|^2\Delta x_i$. It is intuitively clear that the counting rate (density) for φ at $\boldsymbol{x}_i^T$ is equal to the counting rate (density) for $\varphi^{T^{-1}}$ at $\boldsymbol{x}_i$, i.e.,

(Number of detector counts for state φ around $R\boldsymbol{x}$)

$$\begin{aligned} &= |\langle(R\boldsymbol{x})_i|\varphi\rangle|^2 \equiv \left|\langle\boldsymbol{x}_i^T|\varphi\rangle\right|^2 = |\langle\boldsymbol{x}_i|\varphi^{T^{-1}}\rangle|^2 \\ &= (\text{Number of counts for state } \varphi^{T^{-1}} \text{ around } \boldsymbol{x}_i) \end{aligned} \tag{12}$$

In general for the Born probability of an observable $|\psi\rangle\langle\psi|$ in the state φ one has the symmetry condition,

(Probability for observable ψ^T in the state φ)

$$\begin{aligned} &\equiv \mathcal{P}_\varphi(|\psi^T\rangle\langle\psi^T|) \equiv |\langle\psi^T|\varphi\rangle|^2 = |\langle\psi|\varphi^{T^{-1}}\rangle|^2 = \mathcal{P}_{\varphi^{T^{-1}}}(\psi) \\ &= (\text{Probability for observable } \psi \text{ in the state } \varphi^{T^{-1}}). \end{aligned} \tag{13}$$

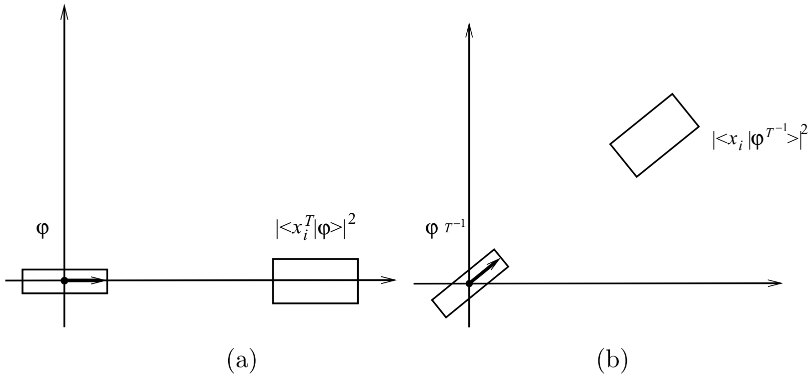

Fig. 1. The symmetry transformation. (a) The symmetry transformation of the detector changes the observable. Now the detector measures the probability of φ at the transformed position x_i^T: $|\langle x_i^T|\varphi\rangle|^2\Delta^3 x$. (b) The symmetry transformation of the preparation apparatus changes the state. Now the detector measures the probability of the transformed state $\varphi^{T^{-1}}$ at the position x_i: $|\langle x_i|\varphi^{T^{-1}}\rangle|^2\Delta^3 x$.

As a consequences of this symmetry condition (13) and under the following two assumptions,

1. the standard axiom of quantum mechanics

$$\{\text{set of states } \varphi\} = \{\text{set of observables } \psi\} = \Phi = \mathcal{H}, \tag{14}$$

and

2. for every transformation $T = T(R)$ of the observable relative to the state

$$T : \psi \to \psi^T, \tag{15a}$$

there exists an inverse transformation *also of the observables*

$$T^{-1} : \psi \to \psi^{T^{-1}}, \tag{15b}$$

one obtains the standard result (Wigner theorem) (Bargmann, 1964; Wigner, 1952):

$$\psi^T = \mathcal{U}(R)\,\psi, \qquad \psi^{T^{-1}} = \mathcal{U}(R^{-1})\,\psi \tag{16}$$

and $\mathcal{U}(R) = \mathcal{U}^\dagger(R^{-1})$ is a unitary representation of the space-time symmetry group (projective representations for Galilei).

The second condition (15a), (15b) is usually not explicitly stated because one does not distinguish between the set of observables $\{\psi\}$ and the set of states $\{\varphi\}$ in standard quantum mechanics. The two assumptions (14) and (15a), (15b) are thus interrelated: If Eq. (14) is assumed then Eq. (16) follows. In particular, using the boundary condition (14) for the dynamical equation (e.g., the Heisenberg equation) it follows directly (Stone, 1932; von Neumann, 1932) that the time translation for the observable is given by

$$\psi(t) = \mathcal{U}(t)\,\psi = e^{iHt}\psi, \quad -\infty < t < \infty. \tag{17a}$$

Similarly, in the Schrödinger picture the unitary group evolution,

$$\varphi(t) = \mathcal{U}^{\dagger}(t)\,\varphi = e^{-iHt}\varphi, \quad -\infty < t < \infty, \tag{17b}$$

follows by the Stone–von Neumann theorem (Stone, 1932; von Neumann, 1932). Conversely, if Eq. (15a) does not hold there will be a conflict with axiom (14). Therefore the question arises: is there for every transformation of the observable ψ also an inverse transformation? For the space translations this is obviously the case. For every forward translation,

$$\boldsymbol{x}_0 \to \boldsymbol{x}_0 + \boldsymbol{x}, \qquad \psi \to \mathcal{U}(\boldsymbol{x})\,\psi, \tag{18a}$$

there is also a backward translation:

$$\boldsymbol{x}_0 \to \boldsymbol{x}_0 - \boldsymbol{x}, \qquad \psi \to \mathcal{U}(-\boldsymbol{x})\,\psi = \mathcal{U}^{-1}\,\psi. \tag{18b}$$

The same holds for rotations, Galilei, and homogeneous Lorentz transformations. For these subgroups there is an answer to the question: what is the probability for the backward translated observable $\mathcal{U}^{-1}\,\psi = \mathcal{U}(-\boldsymbol{x})\,\psi$? Namely it is proportional to

$$\mathcal{P}_{\varphi}(\mathcal{U}^{-1}\,\psi) = |\langle\mathcal{U}^{-1}\,\psi|\varphi\rangle|^2 \sim \text{number of detector counts at } (\boldsymbol{x}_0 - \boldsymbol{x}). \tag{19}$$

Now let us consider the time translation of non-relativistic space-time:

$$t_0 \to t_0 - t, \qquad \psi \to \mathcal{U}(-t)\,\psi \quad \text{for a } t > 0. \tag{20}$$

Is there also an answer to the question: what is the probability for an observable at an arbitrary negative time $(-t)$?

The answer is no! The reason is *causality* in the following manifestation: A state needs to be prepared first, at a time $t = t_0$ (preparation time), before an observable can be measured in it. Only for times $t > t_0$ can the detector count the decay products of a state prepared at $t = t_0$ (possible detector counts before $t = t_0$ would be dismissed as noise). Therefore it makes no sense to expect to measure the probability of an observable $\psi(t)$ in a state φ, $|\langle\psi(t)|\varphi\rangle|^2 = |\langle\mathcal{U}(t)\psi|\varphi\rangle|^2$, for $t \leq t_0$. The probability for the time translated observable $\psi(t)$ in the state φ:

$$\mathcal{P}_{\varphi}(\psi(t)) = |\langle\psi(t)|\varphi\rangle|^2 = |\langle e^{iHt}\psi|\varphi\rangle|^2 = |\langle\psi|e^{-iH^{\times}t}\varphi\rangle|^2 \tag{21}$$

is defined only for

$$t \geq t_0, \quad t_0 = \text{preparationtimeofthestate } \varphi \tag{22}$$

Consequently, the time translated state (Schrödinger picture)

$$\varphi(t) = e^{-iHt}\varphi \quad \text{exist physically only for } t > t_0 = 0. \tag{23a}$$

Therefore it should be defined mathematically only for $t > 0$. Equivalently for the reason of causality (22), (in the Heisenberg picture) the time translated observable

$$\psi(t) = e^{iHt}\psi \quad \text{exists physically only for } t > t_0 = 0. \tag{23b}$$

The mathematical theory has to reproduce these facts. This means the set of states and the set of observables must not fulfill the standard axiom of quantum mechanics (14) since that leads to Eqs. (17a) and (17b). One needs a new hypothesis of quantum theory that predicts the causal time evolution (22), (23a), and (23b).

Experimentally one distinguishes between the states (which are experimentally defined by the preparation apparatus, accelerator) and the observables (which are experimentally defined by the detector, registration apparatus). One has to do it also in the theory. The new hypothesis which distinguishes theoretically between states and observables is the following axiom:

The set of prepared states $\{\varphi^+\}$ defined by preparation apparatus (e.g., in-states) is described by the Gel'fand triplet: $\{\varphi^+\} = \Phi_- \subset \mathcal{H} \subset \Phi_-^\times$. (24a)

The set of observables $\{\psi^-\}$ defined by registration apparatus (e.g., in-observables) is described by the Gel'fand triplet: $\{\psi^-\} = \Phi_+ \subset \mathcal{H} \subset \Phi_+^\times$. (24b)

Here $\Phi_\mp$ are different (dense) Hardy subspaces of the same Hilbert space, $\mathcal{H}$ (Bohm *et al.*, 1997; Bohm, 1981). As a consequence of the Paley-Wiener theorem [Appendix (Bohm *et al.*, 1997; Bohm, 1981) and references thereof].

In these spaces Φ_- and Φ_+ the Schrödinger and the Heisenberg equation integrate to semigroup solutions:

$$\begin{aligned} i\hbar\frac{\partial\varphi^+(t)}{\partial t} &= H\varphi^+(t), \quad \varphi^+ \in \Phi_- \\ &\Longrightarrow \varphi^+(t) = e^{-iHt}\varphi^+ \equiv \mathcal{U}_-^\dagger(t)\varphi^+, \qquad \text{for } 0 \le t < \infty \end{aligned} \tag{25a}$$

$$\begin{aligned} i\hbar\frac{\partial\psi^-(t)}{\partial t} &= -H\psi^-(t), \quad \psi^- \in \Phi_+ \\ &\Longrightarrow \psi^-(t) = e^{iHt}\psi^- \equiv \mathcal{U}_+(t)\psi^-, \qquad \text{for } 0 \le t < \infty \end{aligned} \tag{25b}$$

In the same way as the Hilbert space is realized by the space of Lebesgue square integrable function, the two Hardy space $\Phi_\mp$ can also be defined as the spaces of (their) energy wave functions. The space Φ_- is defined (is mathematically "realized") by the energy wave functions $\varphi^+(E) = \langle^+ Ejj_3|\varphi^+\rangle$ which are Hardy functions analytic in lower complex energy plane. The space Φ_+ is defined by the energy wave functions $\psi^-(E) = \langle^- Ejj_3|\psi^-\rangle$ which are Hardy and analytic in the upper complex energy plane. The kets $|Ejj_3\,^\pm\rangle \in \Phi_\mp^\times$ are the (mathematically defined) Lippmann-Schwinger kets $|E^\pm\rangle \equiv |E \pm i\epsilon^\pm\rangle$ representing the in- and out- going plane waves. [Remark regarding notation: since, by hypothesis (24a), $\langle^+ E|\varphi^+\rangle$ is Hardy in the lower complex semi plane $\mathbf{C}_-$, $\langle\varphi^+|E + i\epsilon^+\rangle = (\langle^+ E + i\epsilon|\varphi^+\rangle)^*$ is Hardy in the upper complex semi plane, i.e., for $E + i\epsilon$; and similar for $|E - i\epsilon^-\rangle$. The mismatch between the notations for vectors φ^+ and for spaces $\Phi_- = \{\varphi^+\}$ is due to the mismatch of the notation in physics of scattering theory (for in-states φ^+) and the notation in mathematics of Hardy spaces for Φ_-, analytic in $\mathbf{C}_-$.]

The distinction between the axiom (14) of standard quantum mechanics and the new Hardy space axiom (3.) is that according to Eq. (14) the energy wave functions of state and observables are (classes of Lebesgue-) square integrable functions. In most physics problems one does not worry much about the Lebesgue integrability but instead wants Dirac kets. Then one replaces the hypothesis (14) by a slightly stronger condition on the set of states $\{\varphi\}$:

$$\{\varphi\} = \{\psi\} = \Phi \subset \mathcal{H} \subset \Phi^\times. \tag{26}$$

Here the energy wave functions are Schwartz space functions $\{\varphi(E)\} = \{\psi(E)\} = \mathcal{S}$, and the Dirac kets $|E\rangle \in \Phi^\times$ are defined as functionals on the Schwartz space.

Integrating the Schrödinger and Heisenberg equations under the boundary conditions (26) would still lead to the group evolution (Wickramasekara and Bohm, 2002).

The Lippmann-Schwinger kets $|E \pm i\epsilon^\pm\rangle$, because of the infinitesimal imaginary part $\pm i\epsilon$, cannot be ordinary Dirac kets (Schwartz space functionals)—though this is always tacitly assumed, also for the relativistic case (Weinberg, 1995). The time asymmetric boundary conditions contained in the Lippmann–Schwinger (integral-) equations do not admit unitary group evolutions.

The energy wave functions of the new hypothesis (3.) are not only smooth and rapidly decreasing, but they are also boundary values of analytic functions in the upper (for $\{\psi^-(E)\}$) and lower (for $\{\varphi^+(E)\}$) complex semi plane (of the second or lower sheet of the S-matrix). The dual spaces $\Phi_\pm^\times$ do not only contain the Lippmann-Schwinger kets $|E \pm i\epsilon^\pm\rangle \in \Phi_\mp^\times$ and their analytic continuation into the complex semi planes, but they contain also generalized eigenvectors of the self adjoint Hamiltonians with eigenvalues that belong to the resonance poles (Bohm *et al.*, 1997; Bohm, 1981) and those that, e.g., belong to higher order poles and which come in Jordan blocks (Antoniou *et al.*, 1998; Bohm *et al.*, 1997; Baumgärtel, 1984). The importance of this axiom is: it provides a consistent and unified theory of resonances and decay. For this the analyticity and Hardy space property in energy E is needed (Bohm *et al.*, 1997; Bohm, 1981).

In non relativistic case this theory is like the Weisskopf-Wigner approximation *with* continuum term. In the relativistic case one replaces the energy E by the invariant energy squared $s = p^{\mu} p_{\mu}$ and obtains a relativistic resonance defined by the S-matrix pole $s_R = (M_R - i\Gamma_R/2)^2$ (Bohm *et al.*, 2003).

As a consequence of the Hardy space analyticity in E all the vectors have a semigroup time evolution like our empirical conclusion Eqs. (23a) and (23b). The semigroup for the elements of $\Phi_+^{\times}$ is given by the conjugate operators $\mathcal{U}_+^{\times}(t)$ of the operator

$$\mathcal{U}_+(t) \equiv \mathcal{U}(t)|_{\Phi_+} = e^{iHt}|_{\Phi_+} = e^{iH_+t}. \tag{27}$$

This conjugate operator is a uniquely defined extension of the Hilbert space adjoint operator $\mathcal{U}^{\dagger}(t)$ to the larger space $\Phi_+^{\times} \supset \mathcal{H}$:

$$\mathcal{U}_+^{\dagger}(t) \subset \mathcal{U}_+^{\times}(t) = (e^{iH_+t})^{\times} = e^{-iH_+^{\times}t}. \tag{28}$$

Here $\mathcal{U}_+$ and H_+ denote the restrictions of the operators $\mathcal{U}$ and H in the space $\mathcal{H}$ to the (dense) subspace Φ_+, and $H_+^{\times}$ is the conjugate operator of $H_+ = H|_{\Phi_+}$ [12].

Similarly for the elements of $\Phi_-^{\times}$ the conjugate operator $\mathcal{U}^{\dagger\times}(t)$ of the operator

$$\mathcal{U}_-^{\dagger}(t) \equiv \mathcal{U}(-t)|_{\Phi_-} = e^{-iH_-t} \tag{29}$$

is given by as the unique extension of the Hilbert space operator $\mathcal{U}(t)$ to $\Phi_-^{\times} \supset \mathcal{H}$:

$$\mathcal{U}(t) = \mathcal{U}^{\dagger\dagger}(t) \subset \mathcal{U}_-^{\dagger\times}(t) = (e^{-iH_-t})^{\times} = e^{iH_-^{\times}t}. \tag{30}$$

Here $\mathcal{U}_-^{\dagger}(t)$ and $H_- = H|_{\Phi_-}$ denote the restrictions of the Hilbert space operators $\mathcal{U}^{\dagger}(t)$ and H to the dense subspace Φ_-.

The Hardy space axiom thus accomplishes a dual purpose:

1. it provides the analyticity properties in energy that are needed to obtain a consistent theory of resonances and decay,
2. it leads to the semigroup evolution needed for the causality condition.

The property of time evolution extends to Galilei transformations. Complex energy and time asymmetry are thus complementary aspect of one and the same theoretical hypothesis. That these two diverse phenomena like resonance and decay on the one hand and causality on the other have the same theoretical basis is remarkable. Since the time-translation subgroup is represented by semigroup of operators in the space of observables Φ_+,

$$\mathcal{U}_+(1, t, 0, 0) = \mathcal{U}_+(t) = e^{iH_+t}, \tag{31}$$

and the space of states Φ_-,

$$\mathcal{U}_+^{\dagger}(1, -t, 0, 0) = \mathcal{U}_+^{\dagger}(t) = \mathcal{U}(-t)|_{\Phi_-} = e^{-iH_-t}, \tag{32}$$

the Galilei group of non relativistic space-time is represented in the space of observables by the semigroup $\mathcal{U}_+(R, t, \boldsymbol{x}, \boldsymbol{v})$, $t \geq 0$ and in the space of states by the semigroup $\mathcal{U}_-(R, -t, \boldsymbol{x}, \boldsymbol{v})$, $t \geq 0$.

The same holds for the relativistic space-time (Bohm *et al.*, 2003). The transformations of the detected observables relative to the prepared state form only a semigroup into the forward light cone,

$$\mathcal{P}_+ = \{(\Lambda, x) : (\Lambda^0{}_0 \geq 1, \det\Lambda = +1),\ \ x^2 = t^2 - \boldsymbol{x}^2 \geq 0,\ \ t \geq 0\}. \tag{33}$$

The physical interpretation of this restriction to the semigroup $\mathcal{P}_+$ is the following: Let

$$|\langle\psi|[m^2, j]\hat{\boldsymbol{p}} j_3{}^-\rangle|^2 \tag{34}$$

be the probability density to detect the observable ψ (decay product) in the "generalized" momentum eigenstate $|[m^2, j]\hat{\boldsymbol{p}} j_3{}^-\rangle$ (Wigner basis ket). Then

$\langle\mathcal{U}_+(\Lambda, x)\psi|[m^2, j]\hat{\boldsymbol{p}} j_3{}^-\rangle$ would be the probability amplitude for the transformed observable $\mathcal{U}_+(\Lambda, x)\psi$ in the same state $|[m^2, j]\hat{\boldsymbol{p}} j_3{}^-\rangle \in \Phi_+^\times$.

This is equal to

$\langle\psi|\mathcal{U}_+^\times(\Lambda, x)|[m^2, j]\hat{\boldsymbol{p}} j_3{}^-\rangle$ the probability amplitude for ψ in the transformed "generalized" state $\mathcal{U}_+^\times(\Lambda, x)|[m^2, j]\hat{\boldsymbol{p}} j_3{}^-\rangle \in \Phi_+^\times$.

The restriction of the transformations (of observable relative to state) to semigroup transformations of the forward light cone $\mathcal{U}_+(\Lambda, x)$, $(\Lambda, x) \in \mathcal{P}_+$ means

1. $t \geq 0$: a state must be prepared first (at $t = 0$) before one can speak of probabilities for observables (causality),
2. $x^2 = t^2 - \boldsymbol{x}^2 \equiv t^2 - r^2/c^2 \geq 0$ or $t^2 \geq r^2/c^2$: Born probabilities ("the signal") can only propagate with a velocity r/t which is smaller than the velocity of light, $r/t \leq c$ (Einstein causality).

Semigroup representations $[s_R, j]$ for relativistic quasistable particles were identified by Schulman (1970) in his classification of irreducible representations of $\mathcal{P}_+$. They were also derived from the resonance pole position at $s_R = (M_R - i\Gamma_R/2)^2$ of the j-th partial S-matrix $S_j(s)$ (Bohm *et al.*, 2003).

4. SUMMARY AND CONCLUSIONS

Using causality arguments we found that quantum mechanical symmetry transformations of the observable relative to the state ($\mathcal{U}_+(t)$ (non relativistic) and $\mathcal{U}_+(\Lambda, x)$ (relativistic)) do not form a group but only a *semigroup*. This is in conflict with the consequences of the standard axiom (14) of quantum mechanics. Experimentally one always distinguishes between states (preparation apparatus, accelerator) and observables (registration apparatus, detector). If one also distinguishes mathematically between states and observables, one is led to the new axiom (3.). From the new hypothesis (3.) one derives semigroup representations of space-time transformations, e.g., $\psi^-(t) = e^{iHt}\psi^-$ for $t \geq 0$. This introduces a new concept, the semigroup time $t_0 = 0$, whereas standard quantum mechanics does not distinguish any particular time. What is the meaning of this *beginning of time* t_0? Why have we not been more aware of this t_0?

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